A matter of notations and conventions.[1]

Conventions and notations (Deja Vu). Some of them throw light over the underlying patterns to a particular problem. These same make darker the things in different problems. Therefore it becomes also a matter of mathematical maturity and good taste the ability of picking a proper representation.

Something that might be taken for granted here:
In all the cases the simplicity is a double-edged sword!.

Motivation: Given the first three natural numbers $1, 2, 3$, when they are used as indices for some expression involving tensors (vectors, matrices, ...), for those situations such that there are necessary only these three numbers, we might define a parametrization rule among them, like this one:

By ranking two of them when the third is excluded. This is (in words): Given the first three natural numbers, which of them is the min? when j is excluded (here j can be take the values $1, 2$ or 3 ONLY). We might ask the same for the max. Then this lead us to a matrix composed by 3 permutations in lexicographic ascending order in the naturals:

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

The first column of such matrix have assigned the special meaning of being the number to be excluded before the rank operation.

Such matrix was accidentally observed at least by the author (closing 2007) but only recently understood as a very particular case of an useful strategy to deal with permutations in the naturals arranged in lexicographic ascending order. The particular calculation where such 3×3 matrix arisen was:

[1]By: R. J. Cano, on Feb 2014. Explanation and motivation for the origin (since 2007) and the submission to OEIS of the sequence $A237265$. E-mail: remy@ula.ve

The vector product between two vectors in the Euclidean space and Cartesian coordinates,

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

After a proper change of representation. By renaming: $\overrightarrow{A} = \overrightarrow{V}_1$, $\overrightarrow{B} = \overrightarrow{V}_2$, and for example: $\hat{\imath} = \overrightarrow{e}_1$, $\hat{\jmath} = \overrightarrow{e}_2$, $\hat{k} = \overrightarrow{e}_3$ or any other of the six possible bijections between these notations for the basis, becoming:

$$\overrightarrow{V}_1 \times \overrightarrow{V}_2 = \begin{vmatrix} \overrightarrow{e}_1 & \overrightarrow{e}_2 & \overrightarrow{e}_3 \\ V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \end{vmatrix}$$

Where it is now observable from:

$$\overrightarrow{V}_1 \times \overrightarrow{V}_2 = \begin{vmatrix} V_{12} & V_{13} \\ V_{22} & V_{23} \end{vmatrix} \overrightarrow{e}_1 - \begin{vmatrix} V_{11} & V_{13} \\ V_{21} & V_{23} \end{vmatrix} \overrightarrow{e}_2 + \begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} \overrightarrow{e}_3$$

That we might rename once more,
$\overrightarrow{e}_k = \overrightarrow{V}_{0k}$ for the basis (with $k = 1, 2,$ or 3), obtaining:

$$\overrightarrow{V}_1 \times \overrightarrow{V}_2 = \begin{vmatrix} V_{12} & V_{13} \\ V_{22} & V_{23} \end{vmatrix} \overrightarrow{V}_{01} - \begin{vmatrix} V_{11} & V_{13} \\ V_{21} & V_{23} \end{vmatrix} \overrightarrow{V}_{02} + \begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} \overrightarrow{V}_{03}$$

And the enumeration $0, 1, 2$ present there in the indices for the row becomes $1, 2, 3$, with a shift by $+1$. Therefore, the previous expression can be considered identical in structure to:

$$\overrightarrow{V}_2 \times \overrightarrow{V}_3 = \begin{vmatrix} V_{22} & V_{23} \\ V_{32} & V_{33} \end{vmatrix} \overrightarrow{V}_{11} - \begin{vmatrix} V_{21} & V_{23} \\ V_{31} & V_{33} \end{vmatrix} \overrightarrow{V}_{12} + \begin{vmatrix} V_{21} & V_{22} \\ V_{31} & V_{32} \end{vmatrix} \overrightarrow{V}_{13}$$

Where everything would be regarded as expressed in the basis $\overrightarrow{V}_{0k}$ (with k in $1, 2, 3$). There is nothing new of course, the underlying mathematical concepts are the same: Permutations and the determinant of a matrix.

Let us consider a determinant instead of a vector product:

$$det\left(\tilde{V}\right) = \begin{vmatrix} V_{22} & V_{23} \\ V_{32} & V_{33} \end{vmatrix} V_{11} - \begin{vmatrix} V_{21} & V_{23} \\ V_{31} & V_{33} \end{vmatrix} V_{12} + \begin{vmatrix} V_{21} & V_{22} \\ V_{31} & V_{32} \end{vmatrix} V_{13}$$

Given $\tilde{A}$ in the way already defined and since $k = A_{k1}$ for $k = 1, 2$ or 3, if we now adopt additionally the notation $\dot{k} = A_{k2}$ and $\ddot{k} = A_{k3}$, the previous determinant might be expressed briefly as follows:

$$det\left(\tilde{V}\right) = \sum_{k=1}^{3} (-1)^{(k+1)} V_{1k} \begin{vmatrix} V_{2\dot{k}} & V_{2\ddot{k}} \\ V_{3\dot{k}} & V_{3\ddot{k}} \end{vmatrix}$$

Btw, the vector product of two vectors $\overrightarrow{V}_2$ and $\overrightarrow{V}_3$ both expressed in the basis given by the vectors $\overrightarrow{V}_{1j}$ (with $j = 1, 2, 3$).

$$\overrightarrow{V}_2 \times \overrightarrow{V}_3 = \sum_{k=1}^{3} (-1)^{(k+1)} \begin{vmatrix} V_{2\dot{k}} & V_{2\ddot{k}} \\ V_{3\dot{k}} & V_{3\ddot{k}} \end{vmatrix} \overrightarrow{V}_{1k}$$

Perhaps much more easy to recall than another related formulas.

Moreover[2], it is possible to generalize the same kind of parametrization for determinants of matrices greater than 3×3, leading in a natural way to the contemplation of $A237265$.

There in the previous context, the reader might notice that for any allowed k:

$$k + \dot{k} + \ddot{k} = k\dot{k}\ddot{k} = 3! = 6$$

Another interesting fact might be also, the existing connection between those sets of m permutations like the ones in $\tilde{A}$, and the whole sets of $m!$ permutations in lexicographic ascending for the first m naturals, revealing the iterative algorithm attributed to Narayana Pandita (Please see references in A237265) as its own parallel version.

[2]See: `http://oeis.org/w/images/d/dc/PartitionPermutations.txt` for a PARI-GP example.